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▶ **To cite this version:**

Yousra Chabchoub, Christine Fricker, Hanene Mohamed. Analyzing a Bloom Filter Algorithm via a join-the-shortest-queue queuing system. 2009. hal-00465304

HAL Id: hal-00465304

https://hal.science/hal-00465304

Preprint submitted on 26 Mar 2010

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Noname manuscript No.
(will be inserted by the editor)

Analyzing a Bloom Filter Algorithm via a join-the-shortest-queue queuing system

Yousra Chabchoub · Christine Fricker · Hanene Mohamed

Received: date / Accepted: date

Abstract This paper deals with the problem of identifying elephants in the Internet Traffic. The aim is to analyze a new adaptive algorithm based on a Bloom Filter. This algorithm uses a so-called min-rule which can be described as in the supermarket model. This model consists of joining the shortest queue among d queues selected at random in a large number of m queues. In case of equality, one of the shortest queues is chosen at random. An analysis of a simplified model gives an insight into the error generated by the algorithm for the estimation of the number of the elephants. The main conclusion is that, as m gets large, there is a deterministic limit for the empirical distribution of the filter counters. Limit theorems are proved and the limit is identified. It depends on key parameters. The condition for the algorithm to perform well is discussed. Theoretical results are validated by experiments on a traffic trace from France Telecom and by simulations.

Keywords Elephant and mice: networking mining · Bloom Filter · Supermarket model

1 Introduction

To be efficient, network traffic measurement methods have to be adapted to the actual traffic characteristics. Internet links currently carry a huge amount of data at a very high bit rate (40 Gb/s in OC-768). To analyze on-line this traffic, scalable algorithms are required. They have to operate fast, using a limited small memory. The traffic is

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mainly analyzed at the *flow* level. A flow is a sequence of packets defined by the classical 5tuple composed of the source and destination addresses, the source and destination port numbers together with the protocol type. Flow statistics are very useful for traffic engineering and network management. In particular, information about large flows (also called elephants) is very interesting for many applications. Note that an elephant is a flow with at least K packets, where K is in practice equal to 20. Elephants are not numerous (around 5 to 20% of the number of flows), but they represent the main part (80-90%) of the traffic volume in terms of packets. Elephant statistics can be exploited in various fields such as attack detection or accounting. In the literature, some probabilistic algorithms have been developed to estimate on-line the number of elephants in a dense traffic. In [7], Flajolet analyzed the adaptive sampling algorithm proposed by Wegman. This algorithm is based on a special sampling method that provides a random set of the original flows. Some characteristics on elephants (number, size distribution) can be inferred from this sample. Some other algorithms (see [8, 2]) based on sampling are designed to provide elephant statistics, but they seem to be designed to very specific flow size distribution or require an a priori knowledge of the total number of flows to recover the loss of information caused by the sampling. Moreover none of these algorithms are able to identify elephants, i.e., to give their addresses. Such information is particularly useful for attack detection. For this application, an attack, seen as an elephant in some sense to define, must be detected even if it is alone, and the previous algorithms, based on sampling, fail to do this.

For that, the paper deals with an algorithm based on Bloom filters. Estan and Varghese [6] proposed a first version. It is quick enough and it uses a limited memory, but it is not adapted to traffic variations. It uses a fixed parameter which should be adjusted according to traffic intensity. Azzana in [1] proposes an improvement for this algorithm by adding a refreshment mechanism that depends on traffic variations. The principle of this latter algorithm is the following. The filter is composed of d stages. Each stage contains m counters and is associated to a hashing function. When a packet is received, its IP header is hashed by the d independent hashing functions and the corresponding counter in each stage is incremented by one. When each counter reaches K (the smallest elephant's size), the corresponding flow is considered as an elephant. Due to the heavy Internet traffic, the filter needs to be sometimes refreshed, otherwise all the counters will exceed the threshold K, and then all the flows will be seen as elephants. The idea is to decrease all counters by one every time the proportion of non null counters reaches a given threshold r. In this way, the refreshment frequency of the filter depends closely on the actual traffic intensity. Notice that the algorithm uses an improvement, the *min-rule*, also called *conservative update* in [6]. It consists in incrementing only the counters among d having the minimum value, for an arriving packet. Indeed, because of collisions, the flow size is greater than or equal to the smallest associated counters, in the ideal configuration where no packet is lost because of the refreshment mechanism. So the min-rule tends to reduce the overestimation of flow size.

Chabchoub *et al.* present in [3] a theoretical analysis of the algorithm proposed by Azzana and described above. Their objective is to estimate the error generated by the algorithm for the estimation of the number of elephants. The analytical study does not take into account the min-rule.

In this paper, we focus on the analysis of the min-rule. For this purpose, in order to obtain a model more standard to analyze, the algorithm described above has been slightly modified. We consider now just one filter and d hashing functions. (Keep in mind the case $d = 2$). An arriving packet increments the smallest counter among the d associated counters. In case of equality, only one counter is incremented at random. In this way, every packet increments exactly one counter. A flow is declared as an elephant when its smallest associated counter reaches $C = K/d$. The same refreshment mechanism is maintained with a threshold r of about 50%. The basic idea is that when the filter is not overloaded, in general, for each arriving packet of a given flow, one of the d counters will be incremented in an alternative way. It means that the d counters will have almost the same values and when the smallest one reaches C, the corresponding flow has a total size of about $K = dC$ packets (C packets hitting each counter).

The advantage of this new algorithm is that each arriving packet increments exactly one counter. In this case, the way to increment the counters is exactly, in a system of m queues, the way a customer joins the shortest queue among d queues chosen at random, ties being solved at random. This model, called *supermarket model* by Mitzenmacher [12] and Luczak and McDiarmid [11], also known as a load-balancing model, or model *with choice*, has been extensively studied in the literature because of its numerous useful applications. In computer science, the central result is stated in a pioneer paper by Azar *et al.* [15], then by Miztenmacher [12], and, by other arguments, by Luczak and McDiarmid [11] for a discrete time model where n balls are thrown into n urns with the choice. It is proved that, with probability tending to 1 as n gets large, the maximum load of an urn is $\log n/\log\log n + O(1)$ when $d = 1$ and $\log\log n/\log d + O(1)$ if $d \geq 2$. Luczak and McDiarmid, in continuous time related models with the choice, explore the concentration of the maximum queue length (see [11], [10]). But the model had also already been studied in Mitzenmacher [12], Vvedenskaya *et al.* [13], Graham [9] and others for mean-field limit theorems. In [12] and [13], a functional law of large numbers is stated: The process of the vector of the tail proportions of queues converges in distribution as m tends to infinity to the unique solution of a differential system. For $d \geq 2$, the differential equation has a unique equilibrium point $u^\rho(k) = \rho^{(d^k-1)/(d-1)}$ for a throughput $\rho < 1$. It means that, when $d \geq 2$, the tail probabilities of the queue length decrease drastically. In Vvedenskaya and Suhov [14], variants of the choice policy and general service time distributions are investigated. Graham in [9] proves the convergence of the invariant measures to the Dirac measure at u^ρ. In other words, when m is large, the stationary vector of the proportions of queues with k customers is essentially deterministic and given by this limit.

The aim of this paper is to analyze the min-rule via the supermarket model and to evaluate the performance of the new proposed algorithm. In particular, we want to calculate the error generated by the algorithm on the estimation of number of elephants. Notice that this error is due to both false negatives (missed elephants) and false positives (mice considered as elephants). There is a trade-off between the proportion of false positives and false negatives. Moreover this trade-off depends on the target applications. For example, for attacks detection, it will be better to have false positives, say suspicious flows, than to miss an elephant, i.e. an attack. The operator can further easily make the difference between anomalies and attacks. On the contrary, in order to count elephants, it is reasonable to avoid having too much false positives because mice are numerous and a small proportion of false positives implies a non negligible error on the number of elephants. Let us first focus on false positives. To be declared as an elephant, a mouse must be hashed to one among d counters greater than C after this operation. So the proportion of such counters is a good parameter to investigate in order to evaluate false positives.

The most part of the paper is the analysis of a simple model where the flows are mice of size one. It is relevant because most of the flows are mice so collisions between flows are mainly due to collisions between mice. It turns out that the probability that a mouse is detected as an elephant is bounded by the probability that a given counter is greater than C just before a refreshment time. Thus the problem reduces to analyze the behavior of the model at the refreshment times. Moreover the transition phase is very short thus the study of the stationary behavior is pertinent.

The key idea of the study is to use the Markovian framework in order to rigorously establish limit theorems and analytical expressions in the stationary regime. The main result is that, as m tends to infinity, the evolution of the model is characterized by a dynamical system which has a unique fixed point $\bar{w}$. The interpretation of $\bar{w}$ as a key quantity in a supermarket model with deterministic service times is discussed. Analytical expressions are given in [3] for $d = 1$ but are more complicated to obtain here for $d \geq 2$.

An objective would be to prove the convergence of the invariant measure of the Markov chain as m tends to $+\infty$ to the Dirac measure $\delta_{\bar{w}}$ at the fixed point $\bar{w}$. In practice, such a result is completely crucial. If it is not true, if the sequence of invariant measures do not converge, the system oscillates with long periods of transition between different configurations (metastability phenomenon). So even if the algorithm performs well during a while, it can reach another state where it can give bad results. This question is partially addressed here. The convergence of the invariant measures is conjectured, due to simulations of the algorithm where such a phenomenon has not been observed. For such a result, a possible technique is the existence of a Lyapounov function which both proves the convergence of the dynamical system to its unique fixed point $\bar{w}$ and the convergence of the sequence of invariant measures to $\delta_{\bar{w}}$. Such a function is exhibited in [3] for $d = 1$ and $C = 2$.

A simulation of the limit distribution $\bar{w}$ is done for a uniform general mice size distribution which is non analytically tractable for $d \geq 2$. It is compared to the experiment on a real trace. This one hour trace is commercial traffic provided by France Telecom. Results are very close. This trace is used as a validation test and no algorithm parameter need to be changed to fit to some characteristics of this particular trace. Experiments have three other goals. First to compare the original version with the min rule to the modified version of the algorithm introduced here. It appears that the latter version exhibits performances as good as the original one. Second, the time between two refreshments is plotted. This quantity is crucial for the trade-off between false positives and false negatives. It depends on the value of the threshold r. If r is high, the counters are high due to collisions, involving many false positives. But refreshment times occur less often so less packets are lost, inducing less false negatives. The analysis shows such a behavior. Experiments validate the analysis. Third, the time to reach the stationary phase is also discussed.

The paper is an extended version of Chabchoub [4]. Sketches of proofs have been replaced by proofs. Experiments have been completed to enlight the algorithm behaviour.

The organization of the paper is as follows: Section II presents the analytical results for the simple model defined to study the question of false positives. Section III is devoted to experiments. Section IV mainly gives a discussion of the way to choose the parameter r in order to have an algorithm which performs well.

2 The Markovian urn and ball model

2.1 Description of the model

In this section, the question of false positives is addressed: The target quantity is the probability for a mouse to be detected by the algorithm as an elephant.

The problem is studied in a simple framework, where flows are reduced to mice of size one. Thus the model can be described as a urn and ball model because one size flows hashed in a filter with m counters can be viewed as balls thrown into m urns with capacity C under the supermarket rule: For each ball, a subset of d urns is chosen at random and the ball is put in the least loaded urn, ties being resolved uniformly. Balls overflowing the capacity C are rejected. Moreover if, after putting the ball, the number of non empty urns exceeds rm, then one ball is removed from every non empty urn.

The probability of a flow to be detected as an elephant is reduced to the probability that, after the ball arrival, all the d chosen urns have C balls. It is bounded by the probability that, just before a refreshment time, after putting the last ball in its urn, all the d urns chosen for that contain C balls. The bound is more convenient to study. Let us focus on the embedded model just before the refreshment times.

2.2 A Markovian framework

For fixed C, let us consider the sequence $(W_n^m)_{n\in\mathbb{N}}$, where W_n^m denotes the vector of the proportions of urns with $0,\dots,C$ balls just before the nth refreshment time. For $m \geq 1$, $(W_n^m)_{n\in\mathbb{N}}$ is an ergodic Markov chain on the finite state space

$$\mathcal{P}_m^{(r)} = \{w \in \left(\frac{\mathbb{N}}{m}\right)^{C+1}, \sum_{i=0}^{C} w(i) = 1,\ \sum_{i=1}^{C} w(i) = \frac{\lceil rm \rceil}{m}\},$$

(where $\lceil rm \rceil$ denotes the smallest integer larger than rm). Thus it has a unique invariant measure π_m.

The problem is that this quantity is combinatorically intractable. Even the transition probability P_m of the Markov chain is awfully difficult to write. Nevertheless, one could expect an asymptotic of this quantity when m gets large. In other words, the limit of the invariant measures π_m when m is large is investigated.

2.3 A dynamical system

The way which is used here to obtain limit theorems is very classical (see [5] for example). In fact, similar results for $d = 1$ can be found in Chabchoub et al [3]. The following results extend the case $d = 1$ to $d \geq 1$. Of course the motivation here is the case $d \geq 2$. The proofs must often be rewritten with new arguments and the sections which are still valid will be in general omitted.

Let us introduce some notations. Let

$$\mathcal{P} \stackrel{def}{=} \{w \in \mathbb{R}_+^{C+1}, \sum_{i=0}^{C} w(i) = 1\}$$
$$\text{and } \mathcal{P}^{(r)} \stackrel{def}{=} \{w \in \mathbb{R}_+^{C+1}, \sum_{i=0}^{C} w(i) = 1 \text{ and } \sum_{i=1}^{C} w(i) = r\}$$

be the state spaces. Let s be the shift defined as

$$s : w \mapsto (w(0) + w(1), w(2), \ldots, w(C), 0) \text{ on } \mathcal{P}$$

and

$$\lambda : \mathcal{P}^{(r)} \to \mathbb{R}^+,\ w \mapsto \int_{r-w(1)}^{r} \frac{du}{1-u^d}. \tag{1}$$

For the vector of proportions $v \in \mathcal{P}$, it is more convenient to deal with the vector of the tail proportions u defined by $u_k = \sum_{i \geq k} v_i$. G is then defined on $\mathcal{P}^{(r)}$ by

$$G(w) = v(\lambda(w)) \tag{2}$$

where $(v(t))$ is associated to $(u(t))$ the unique solution of

$$\frac{du_k}{dt} = u_{k-1}(t)^d - u_k(t)^d,\ k \in \{1, \ldots C\},\ u_0 = 1 \tag{3}$$

with initial condition $u(0)$ corresponding to $v(0) = s(w)$. Notice that G maps, by definition of λ, $\mathcal{P}^{(r)}$ to $\mathcal{P}^{(r)}$. To emphasize the dependence from the initial condition, $G(w)$ will also be denoted by $v(s(w), \lambda(w))$.

The following result is that, as $m \to +\infty$, the Markov chain converges in distribution to a deterministic dynamical system which will be explicited.

Proposition 1 *If W_0^m converges in distribution to $w \in \mathcal{P}_m^{(r)}$ then $(W_n^m)_{n\in\mathbb{N}}$ converges in distribution to the dynamical system $(w_n)_{n\in\mathbb{N}}$ given by the recursion*

$$w_{n+1} = G(w_n),\ n \in \mathbb{N}$$

where G is defined by equation (2).

Proof The result is a consequence of the convergence of the transition P_m of the Markov chain $(W_n^m)_{n\in\mathbb{N}}$ as m tends to $+\infty$ to P given by

$$P(w, .) = \delta_{G(w)}.$$

It means that, starting from w just before a refreshment time, at the next refreshment time, the vector of the proportions of urns tends to $G(w)$ when m tends to $+\infty$. The uniform convergence stated by the following lemma provides the convenient way to prove Proposition 1.

Lemma 1 *For $\varepsilon > 0$,*

$$\sup_{w \in \mathcal{P}_m^{(r)}} P_m(w, \{w' \in \mathcal{P}_m^{(r)} : ||w' - G(w)|| > \varepsilon\}) \underset{m\to+\infty}{\to} 0.$$

Proof The idea of the proof is that, starting from w (with $\lceil rm \rceil$ non empty urns), after refreshment, the vector of the proportions becomes $s(w)$ defined by

$$s(w) = (w(0) + w(1), w(2), \dots, w(C), 0)$$

where the proportion of non empty urns is $r - w(1)$. Then a number τ_1^m of balls are thrown in order to reach a state w' with again $\lceil rm \rceil$ non empty urns. It has to be proved that w' is close to $G(w)$. There are three steps:

Step 1. It can be proved that this number τ_1^m is deterministic at first order when m is large, equivalent to $\lambda(w)m$, where $\lambda(w)$ is defined by equation (1). More precisely,

$$\sup_{w \in \mathcal{P}_m^{(r)}} \mathbb{P}_w \left(\left| \frac{\tau_1^m}{m} - \lambda(w) \right| > \varepsilon \right) \underset{m \to \infty}{\to} 0. \tag{4}$$

By Bienaymé-Chebychev's inequality, it is enough to prove that

$$\sup_{w \in \mathcal{P}_m^{(r)}} \left| \mathbb{E}_w \left(\frac{\tau_1^m}{m} \right) - \lambda(w) \right| \underset{m \to \infty}{\to} 0 \tag{5}$$

and

$$\sup_{w \in \mathcal{P}_m^{(r)}} \mathrm{Var}_w \left(\frac{\tau_1^m}{m} \right) \underset{m \to \infty}{\to} 0. \tag{6}$$

To see it, use that, starting from w, τ_1^m has an analytical expression as a sum of different numbers Y_l of balls necessary to hit the $(l+1)$th non empty urn. Indeed,

$$\tau_1^m = \sum_{l=(d-1) \vee (\lceil rm \rceil - w(1)m)}^{\lceil rm \rceil - 1} Y_l, \tag{7}$$

where the Y_ls for $l \in \mathbb{N}$ are independent random variables with geometrical distributions on $\mathbb{N}^*$ with respective parameters

$$a_l = \prod_{j=0}^{l-1} \frac{l-j}{m-j},$$

i.e. $\mathbb{P}(Y_l = n) = (a_l)^{n-1}(1 - a_l)$, $n \geq 1$. By equation (7), as $\mathbb{E}(Y_l) = 1/(1 - a_l)$, denoting $a \vee b = \sup(a, b)$,

$$\mathbb{E}_w \left(\frac{\tau_1^m}{m} \right) = \frac{1}{m} \sum_{l=(d-1) \vee (\lceil rm \rceil - w(1)m)}^{\lceil rm \rceil - 1} \frac{1}{1 - a_l}.$$

using that, for $l \geq d - 1$,

$$\left(\frac{l - d + 1}{m} \right)^d \leq a_l \leq \left(\frac{l}{m - d + 1} \right)^d$$

it yields

$$\frac{1}{m}\sum_{l=(d-1)\vee(\lceil rm\rceil-w(1)m)}^{\lceil rm\rceil-1}\frac{1}{1-\left(\frac{l-d+1}{m}\right)^{d}}\leq\mathbb{E}_w\left(\frac{\tau_1^m}{m}\right)$$
$$\leq\frac{1}{m}\sum_{l=\lceil rm\rceil-w(1)m}^{\lceil rm\rceil-1}\frac{1}{1-\left(\frac{l}{m-d+1}\right)^{d}}.$$

With a change of index in the left-hand side term and denoting $m'=m-d+1$,

$$\frac{1}{m}\sum_{l=(\lceil rm\rceil-w(1)m-d+1)^+}^{\lceil rm\rceil-d}\frac{1}{1-\left(\frac{l}{m}\right)^{d}}\leq\mathbb{E}_w\left(\frac{\tau_1^m}{m}\right)$$
$$\leq\frac{1}{m'+d-1}\sum_{l=\lceil r(m'+d-1)\rceil-w(1)(m'+d-1)}^{\lceil r(m'+d-1)\rceil-1}\frac{1}{1-\left(\frac{l}{m'}\right)^{d}}.$$

A comparison with integrals leads to the following inequalities: On one hand,

$$\mathbb{E}_w\left(\frac{\tau_1^m}{m}\right)\leq\frac{m'}{m'+d-1}\int_{\frac{\lceil r(m'+d-1)\rceil-w(1)(m'+d-1)}{m'}}^{\frac{\lceil r(m'+d-1)\rceil}{m'}}\frac{dt}{1-t^d}.$$

The right-hand side of this inequality denoted by $f_m(w(1))$ converges for each $w(1)\in[0,r]$ to $\lambda(w)=\int_{r-w(1)}^{r}dt/(1-t^d)$ when m tends to $+\infty$, using that m'/m tends to 1 when m tends to infinity. Moreover, f_m is increasing on $[0,r]$. Thus, by Dini's theorem, f_m converges uniformly on $[0,r]$ as m tends to $+\infty$. On the other hand,

$$\mathbb{E}_w\left(\frac{\tau_1^m}{m}\right)\geq\frac{1}{m}\sum_{l=(\lceil rm\rceil-w(1)m-d+1)^+-1}^{\lceil rm\rceil-d-1}\frac{1}{1-\left(\frac{l+1}{m}\right)^{d}}\geq\int_{\frac{(\lceil rm\rceil-w(1)m-d+1)^+-1}{m}}^{\frac{\lceil rm\rceil-d}{m}}\frac{dt}{1-t^d}.$$

Denoting by $g_m(w(1))$ the previous integral, by the same argument, g_m converges uniformly on $[0,r]$ as m tends to $+\infty$. Using that $w(1)\in[0,r]$ when $w\in\mathcal{P}_m^{(r)}$, it proves (5). For (6), as $\mathrm{Var}(Y_l)=a_l/(1-a_l)^2$,

$$\mathrm{Var}(\tau_1^m)=\sum_{l=\lceil rm\rceil+w(1)m}^{\lceil rm\rceil-1}\frac{a_l}{(1-a_l)^2}$$

Following the same lines,

$$\frac{1}{m}\sum_{l=(\lceil rm\rceil-w(1)m-d+1)^+}^{\lceil rm\rceil-d}\frac{\left(\frac{l}{m}\right)^{d}}{\left(1-\left(\frac{l}{m}\right)^{d}\right)^2}\leq\frac{\mathrm{Var}(\tau_1^m)}{m}$$
$$\leq\frac{1}{m'+d-1}\sum_{l=\lceil r(m'+d-1)\rceil-w(1)(m'+d-1)}^{\lceil r(m'+d-1)\rceil-1}\frac{\left(\frac{l}{m'}\right)^{d}}{\left(1-\left(\frac{l}{m'}\right)^{d}\right)^2}.$$

But, as $t \mapsto t/(1-t)^2$ is increasing on $\mathbb{R}_+$, by the same argument as before,

$$\sup_{w \in \mathcal{P}_m^{(r)}} \frac{\mathrm{Var}(\tau_1^m)}{m} \underset{m\to\infty}{\to} \int_{r-w(1)}^{r} \frac{t^d}{(1-t^d)^2}$$

which proves (6).

Step 2. Using both this first step and Chernoff's bound provide a natural coupling described as follows. For some fixed $w \in \mathcal{P}_m^{(r)}$, consider m urns with empirical distribution $s(w)$ and define the following random variables: for $1 \le i \le m$, $N_i^m(w)$ (respectively $\tilde{N}_i^m(w)$ and $\hat{N}_i^m(w)$) is the number of additional balls in urn i when τ_1^m (respectively $m\lambda(w)$ and a Poisson random variable $\mathcal{P}_{\lambda(w)m}$ with parameter $m\lambda(w)$) new balls are thrown in the m urns. One can construct these variables from the same sequence of balls (i.e. of i.i.d. uniform on $\{1, \dots, m\}$ random variables), meaning that balls are thrown in the same locations for both operations until stopping. This provides a natural coupling for the N_i's, the $\tilde{N}_i$'s and the $\hat{N}_i$'s. Let $j \in \{0, \dots, C-1\}$ be fixed. Given $W_0^m = w$, as $j \le C-1$, the capacity constraint does not interfere and $W_1^m(j)$ can be represented as

$$W_1^m(j) = \frac{1}{m} \sum_{k=0}^{j} \sum_{i \in I_{w,k}^m} 1_{\{N_i^m(w) = j-k\}} \tag{8}$$

where $I_{w,k}^m$ is the set of urns with k balls in some configuration of m urns with distribution $s(w)$. The sum over i is exactly the number of urns that contains k balls after the removing of one ball per urn, and having j balls after new balls have been thrown. Let also define

$$\tilde{W}_1^m(j) = \frac{1}{m} \sum_{k=0}^{j} \sum_{i \in I_{w,k}^m} 1_{\{\tilde{N}_i^m = j-k\}},$$

$$\hat{W}_1^m(j) = \frac{1}{m} \sum_{k=0}^{j} \sum_{i \in I_{w,k}^m} 1_{\{\hat{N}_i^m = j-k\}}.$$

By coupling, on the event $\{W_0^m = w,\ |\tau_1^m/m - \lambda(w)| \le \varepsilon/3\}$, the following is true:

$$\mathrm{card}\{i, N_i^m(w) \neq \tilde{N}_i^m(w)\} \le \frac{\varepsilon}{3} m$$

thus, on the same event,

$$|W_1^m(j) - \tilde{W}_1^m(j)| \le \frac{\varepsilon}{3}. \tag{9}$$

Moreover, for $\delta \in]0, 1/2[$ fixed, on the event $\{W_0^m = w,\ |\mathcal{P}_{\lambda(w)m} - \lambda(w)m| \le m^{\delta+1/2}\}$,

$$\frac{1}{m}\mathrm{card}\{i, \tilde{N}_i^m(w) \neq \hat{N}_i^m(w)\} \le m^{\delta-1/2}$$

thus, for m enough large ($m \ge M_0$),

$$|\tilde{W}_1^m(j) - \hat{W}_1^m(j)| \le \frac{\varepsilon}{3}. \tag{10}$$

To obtain the lemma, it is equivalent to prove that, for each $\varepsilon > 0$,

$$\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(||W_1^m - G(w)|| > \varepsilon\right) \underset{m\to\infty}{\to} 0 \tag{11}$$

where $\mathbb{P}_w(.)$ denotes $\mathbb{P}(.|W_0^m = w)$. Since W_1^m and $G(w)$ are probability measures on $\{0,\dots,C\}$, to get (11), it is sufficient to prove that, for $j\in\{0,\dots,C-1\}$,

$$\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(|W_1^m(j) - G(w)(j)| > \varepsilon\right) \underset{m\to\infty}{\to} 0 \tag{12}$$

which reduces to prove that

$$\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(|W_1^m(j) - G(w)(j)| > \varepsilon, \left|\frac{\tau_1^m}{m} - \lambda(w)\right| \le \frac{\varepsilon}{3}, \left|\mathcal{P}_{\lambda(w)m} - \lambda(w)m\right| \le m^{\delta+1/2}\right) \tag{13}$$

tends to 0 when m is large. Indeed, by the first step,

$$\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(\left|\frac{\tau_1^m}{m} - \lambda(w)\right| > \frac{\varepsilon}{3}\right) \underset{m\to\infty}{\to} 0.$$

Moreover,

$$\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(\left|\mathcal{P}_{\lambda(w)m} - \lambda(w)m\right| > m^{\delta+1/2}\right) \underset{m\to\infty}{\to} 0 \tag{14}$$

because, by standard Chernoff's inequality, for $\eta > 0$,

$$\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(\left|\mathcal{P}_{\lambda(w)m} - \lambda(w)m\right| \ge \eta\lambda(w)m\right) \le \exp\left(-\frac{\eta^2\lambda(w)m}{3}\right).$$

Let $\eta = m^{-1/2+\delta}/\lambda(w)$. By the previous inequality, as $\lambda(w)\in[0,F(r)]$,

$$\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(\left|\mathcal{P}_{\lambda(w)m} - \lambda(w)m\right| \ge m^{\delta+1/2}\right) \le \exp\left(-\frac{m^{2\delta}}{3\lambda(w)}\right) \le \exp\left(-\frac{m^{2\delta}}{3F(r)}\right)$$

thus it gives equation (14). To prove the quantity given by (13) tends to 0 when m is large, it is then sufficient to prove that

$$\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(|\hat{W}_1^m(j) - G(w)(j)| > \varepsilon\right) \underset{m\to\infty}{\to} 0.$$

Indeed, let the event $\{|\tau_1^m/m - \lambda(w)| \le \varepsilon/3, \left|\mathcal{P}_{\lambda(w)m} - \lambda(w)m\right| \le m^{\delta+1/2}\}$ be denoted by A. To see it, it is sufficient to write

$$\begin{aligned}\sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(|W_1^m(j) - G(w)(j)| > \varepsilon, A\right) \le{}& \sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(|\hat{W}_1^m(j) - G(w)(j)| > \frac{\varepsilon}{3}, A\right)\\ &+ \sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(|\hat{W}_1^m(j) - \tilde{W}_1^m(j)| > \frac{\varepsilon}{3}, A\right)\\ &+ \sup_{w\in\mathcal{P}_m^{(r)}} \mathbb{P}_w\left(|W_1^m(j) - \tilde{W}_1^m(j)| > \frac{\varepsilon}{3}, A\right).\end{aligned}$$

Because of (9) and (10), the last two terms of the right-hand side of the previous inequality are null for $m \geq M_0$. Therefore, it is sufficient to prove that

$$\sup_{w \in \mathcal{P}_m^{(r)}} \mathbb{P}_w \left(\| \hat{W}_1^m - G(w) \| > \varepsilon \right) \underset{m \to \infty}{\to} 0. \tag{15}$$

It is the aim of the last step.

Step 3. Let us define the queuing supermarket model without departures. Consider m queues with capacity C filled by a Poisson process of customers with rate m as follows: At each customer arrival, d queues are chosen at random and the customer goes to the lowest loaded queue, to one of them at random if there are several. If the number of customers at the queue is already C, the new customer is rejected.

Let $X^m(t) = (X_i^m(t), 1 \leq i \leq m)$ be the queue length vector process, where $X_i^m(t)$ is the number of customers at time t at queue i. Let $(W^m(t))_{t \in \mathbb{R}^+}$ and $(U^m(t))_{t \in \mathbb{R}^+}$ be the Markov processes defined by

$$W_k^m(t) = \frac{1}{m} \sum_{i=1}^{m} 1_{\{X_i^m(t) = k\}}$$

the proportion of queues with k customers at time t and

$$U_k^m(t) = \frac{1}{m} \sum_{i=1}^{m} 1_{\{X_i^m(t) \geq k\}}$$

the proportion of queues with more than k customers at time t. By standard arguments, it is well-known (see Vvedenskaya et al. [13]) that, if $U^m(0) \to_{m \to \infty} u(0)$, the process $(U^m(t))_{t \in \mathbb{R}^+}$ converges to its fluid limit, which is the deterministic process $(u(t) = (u_k(t), 1 \leq k \leq C)$ given as the unique solution of the differential system

$$\frac{du_k}{dt} = u_{k-1}^d - u_k^d \quad (1 \leq k \leq C), u_0 = 1$$

with initial condition $u(0)$ (see Appendix for a proof) in the following sense: For each $t,\ \varepsilon > 0,\ k \in \{0, \ldots, C\}$,

$$\mathbb{P}(\sup_{0 \leq s \leq t} |U_k^m(s) - u_k(s)| > \varepsilon) \underset{m \to \infty}{\to} 0.$$

Moreover denoting by $u(x,t)$ the solution at t with initial condition x and $U_k^m(x,t)$ with obvious notations, for each $\varepsilon > 0,\ k \in \{0, \ldots, C\}$, using that λ is bounded for $w(1) \in [0, r]$ and s is bounded on $\mathcal{P}_m^{(r)}$,

$$\mathbb{P}(\sup_{w \in \mathcal{P}_m^{(r)}} |U_k^m(s(w), \lambda(w)) - u_k(s(w), \lambda(w))| > \varepsilon) \underset{m \to \infty}{\to} 0 \tag{16}$$

where, by abuse of notation, $s(w)$ denotes the tail vector of $s(w)$. Let $G(w)$ defined by (2) as $u(s(w), \lambda(w))$. Using that, for $j \in \{0, \ldots, C-1\}$, given $W_0^m = w$,

$$\begin{aligned} &\hat{W}_1^m(j) - G(w)(j) = \\ &\quad U_j^m(s(w), \lambda(w)) - u_j(s(w), \lambda(w)) - \left(U_{j+1}^m(s(w), \lambda(w)) - u_{j+1}(s(w), \lambda(w)) \right), \end{aligned}$$

equation (16) straightforwardly implies that, for each $w \in \mathcal{P}_m^{(r)}$,

$$\sup_{w \in \mathcal{P}_m^{(r)}} \mathbb{P}_w \left(\| \hat{W}_1^m - G(w) \| > \varepsilon \right) \leq \mathbb{P} \left(\sup_{w \in \mathcal{P}_m^{(r)}} \| \hat{W}_1^m - G(w) \| > \varepsilon \right) \underset{m \to \infty}{\to} 0.$$

It ends the proof of the lemma.

The argument to obtain Proposition 1 from Lemma 1 is standard and detailed in [3, Proposition 1]. It is omitted here.

2.4 Fixed point of the dynamical system

The function

$$\begin{aligned}\mathcal{P}^{(r)} &\longrightarrow \mathcal{P}^{(r)}\\ w &\longmapsto G(w)\end{aligned}$$

being continuous on the convex compact set $\mathcal{P}^{(r)}$, by Brouwer's theorem, it has a fixed point.

It remains to prove the uniqueness of the fixed point. Recall that, for $d = 1$, the proof is based on the interpretation of the fixed point equation

$$G(w) = w$$

as the equation

$$w = \mu_{\lambda(w)} \tag{17}$$

where μ_λ is the invariant measure of some ergodic Markov chain. This Markov chain is the queue length at the service completion times of a $M/G/1/C$ queue with deterministic service times equal to 1 with arrival rate λ. The proof of the uniqueness of the solution of equation (17) is then based on the coupling argument that, if $\lambda \leq \lambda'$ then μ_λ is stochastically dominated by $\mu_{\lambda'}$ (see [3] for details).

Let us try to extend the argument for the case $d \geq 2$. For that, let us consider the following system. Balls arrive according to a Poisson process with rate λm. They are thrown into m urns with capacity C as follows. Each ball joins the least loaded urn among a subset of d urns, chosen at random. The ties are resolved uniformly. At each unit time, one ball is removed from each non-empty urn. It can be proved as in Proposition 1 that the vector of the proportions of urns with k balls just before time n converges, when m is large, to a dynamical system

$$w_{n+1} = H(w_n)$$

where, for v defined by (3) with initial condition $v(0) = s(w)$,

$$H(w) = v(\lambda).$$

But the argument fails for $d \geq 2$. Assume that $H(w) = w$ is the invariant measure equation of some ergodic Markov chain, i.e., of type $wP = w$ for some transition matrix P. As $H(w) = v(\lambda)$,

$$v(\lambda) = wP.$$

From equation (2.3), $u_1(t)$ can be computed explicitly. It gives that

$$v_0(t) = 1 - F^{-1}(t + F(r - w(1)))$$

where $F(x) = \int_0^x \frac{dt}{1-t^d}$ is a one-to-one map from $[0, 1[$ to its image. It means that, for $d \geq 2$, $v_0(\lambda)$ can not be written as $\sum_{j=0}^{C} P_{0,j}w(j)$. Thus another way to prove it can be found and, at this point, the uniqueness of the fixed point is conjectured for $d \geq 2$.

2.5 Identification of the fixed point

If the capacity is infinite, then the parameter $\lambda(\bar{w})$ is equal to r. In this case, it is simple to have the explicit expression of $\bar{w}_1$, which is a good approximation for the case $C = 20/d$, say for $d = 1$ or 2. It is the purpose of this section.

Assume that $C = +\infty$. By definition,

$$\lambda(\bar{w}) = \int_{r-\bar{w}(1)}^{r} \frac{dt}{1-t^d}$$

and $F(x) = \int_0^x \frac{dt}{1-t^d}$ defines a one-to-one map from $[0,1[$ to its image. Using that $\lambda(\bar{w}) = r$ for $C = +\infty$, it can be rewritten

$$r = F(r) - F(r - \bar{w}(1)),$$

or

$$\bar{w}(1) = r - F^{-1}(F(r) - r). \tag{18}$$

Notice that for $d \in \mathbb{N}$, F has an explicit expression. For $d = 1$, $F(x) = -\log(1-x)$ which leads (see [3]) to

$$\bar{w}(1) = (1-r)(e^r - 1).$$

Moreover, for $d = 2$, $F(x) = \mathrm{argth} x$ and thus

$$\bar{w}(1) = r - \frac{r - \mathrm{th} r}{1 - r\,\mathrm{th} r}. \tag{19}$$

In Figure 1, $\bar{w}(1)$ is plotted for $d = 1,\ 2$.

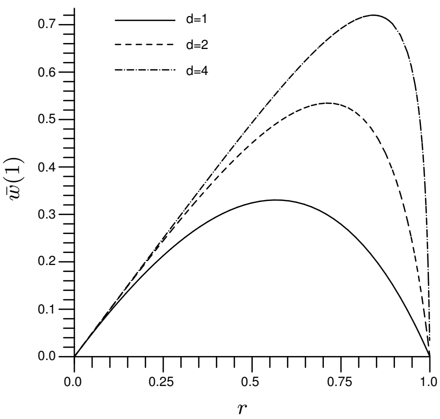

Fig. 1 Limit proportion $\bar{w}(1)$ of counters with value 1 for $d = 1, 2, 4$.

2.6 Convergence of invariant measures

A first result is obtained. It is proved in [3, Proposition 2] and recalled here omitting the proof.

Proposition 2 *Let, for $m \geq 1$, π_m be the stationary distribution of $(W_n^m)_{n\in\mathbb{N}}$. Define P as the transition on $\mathcal{P}^{(r)}$ given by $P(w,.) = \delta_{G(w)}$. Any limiting point π of $(\pi_m)_{m\in\mathbb{N}}$ is a probability measure on $\mathcal{P}^{(r)}$ which is invariant for P i.e. that satisfies $G(\pi) = \pi$.*

As noticed in the introduction, the limiting point of π_m is not necessarily unique, because there is not a unique measure π such that $G(\pi) = \pi$. There is one because G has a unique fixed point thus $G(\delta_{\bar{w}}) = \delta_{\bar{w}}$. But imagine that G has cycles, i.e. that there exist $n \geq 2$ and $w_1, \ldots, w_n$ in $\mathcal{P}^{(r)}$ such that

$$G(w_j) = w_{j+1}\ (1 \leq j < n),\ G(w_n) = w_1$$

then $\pi = 1/n \sum_{j=1}^{n} \delta_{w_j}$ is invariant under G. It gives two different limiting points for π_m. A way to prove the convergence of (π_m) to $\delta_{\bar{w}}$ is to find a Lyapounov function for G (see [3, Theorem 1] for details). Such a Lyapounov function is exhibited in [3] for $d = 1$ and $C = 2$. It is not investigated here.

2.7 General mice size distribution

The subsection deals with the extension of the previous results to a model with general size distribution. An approximated model is taken. Indeed, as mice size are short (with mean close to some units, in real traffic traces, close to 4), an approximated model is to consider that the packets of the mice are thrown without interleaving with packets of other mice in the target counters. It means that the packets of the different mice arrive consecutively in the filter.

The model chosen is thus an urn and ball model where balls are thrown by batches. The balls in a batch are thrown successively, each ball into the least loaded urn among d chosen at random in the m urns. Notice that the set of d urns is the same for every ball of a given batch, but the balls tend to fill these d urns alternatively. The ith batch is composed with S_i balls, where the S_is are independent random variables with distribution denoted by p. The model generalizes the previous one obtained for mice of size one (p(1)=1).

This model, which is simple to analyze in the case $d = 1$ (see [3, Subsection 2.1]) as an extension of the model with mice of size one, becomes much more difficult in the case $d \geq 2$. To our knowledge, the pending queuing system with batch arrivals and customers joining the shortest queue has never been studied.

3 Experiments

In this section, the proposed algorithm is tested against an ADSL commercial traffic trace from France Telecom IP backbone network. This traffic trace has been captured on a Gigabit Ethernet link in October 2003 between 9:00 pm and 10:00 pm. This period corresponds to a peak activity by ADSL customers (the link load was equal to 43.5%),

Table 1 Characteristics of the traffic trace considered in experiments

Traces	Nb. IP packets	Nb. TCP Flows	Duration
FT trace	135 844 423	10 474 665	1 hour

its duration is 1 hour and contains more than 10 millions of TCP flows. Some trace characteristics are given in Table I.

In our experiments, the filter consists of 2^{20} counters ($m = 1.05MB$) associated to two independent hashing functions ($d = 2$). Elephants are here defined as flows with at least 20 packets ($K = 20$).

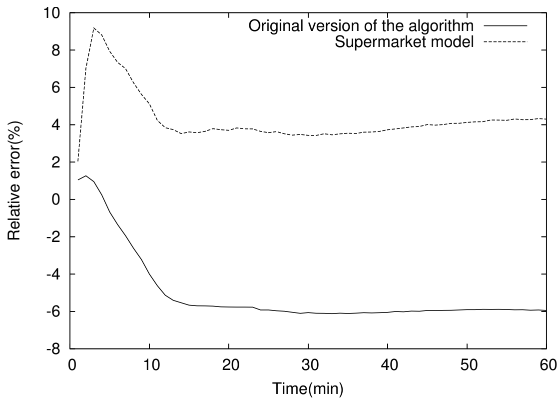

Fig. 2 Impact of the Supermarket model on the estimation of elephants number , $r = 50\%$, France Telecom trace

The relative error on the estimated number of elephants, defined as the difference between the number of detected elephants and the exact number of elephants divided by the exact number of elephants, is plotted in Figure 2. Two different versions of the algorithm are considered: The original algorithm developed in [1] and presented in the introduction and the modified algorithm introduced here. Recall that these two algorithms use the min-rule (incrementing only smallest counters), but in a different way: In case of equality, only one counter is incremented at random in the supermarket scheme, whereas the two counters are incremented in the original version of the algorithm. The proposed alternative algorithm implements the min rule in a more standard way. It allows to exploit well-known analytical results on the supermarket model, which is very studied in the literature. In its original version, the fact that each lowest counter is incremented makes the model more difficult to analyze. Experiments show that both methods give a good estimation of number of elephants, for the whole duration of the trace. Moreover, the supermarket version has less missed elephants, which can be better for some applications.

One can notice that for the modified algorithm, the generated error is higher at the beginning (when time is less than 10 minutes). This can be explained by the fact that starting from an empty filter (with all counters at 0), it takes a long time to reach the filling up threshold (50%) to perform the first refreshment. As a consequence, counters are high compared to their values in the stationary phase. Indeed it has been checked experimentally that, just before the first refreshment, the proportion of counters at C is up to 10 times higher than in the stationary phase.

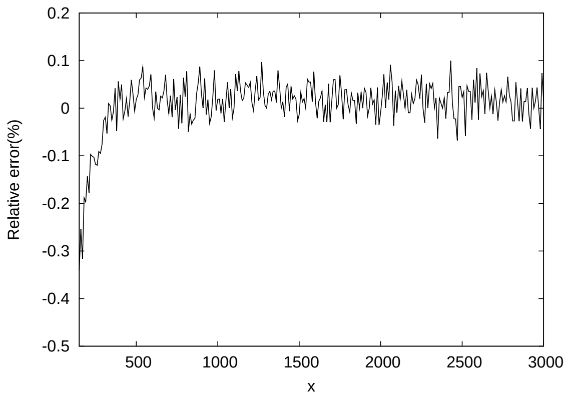

Fig. 3 Relative error on the number of flows with more than x packets, $m = 1.05MB$, $r = 50\%$, France Telecom trace

Once a flow is declared as an elephant by the algorithm, all its new packets are no more inserted into the filter. They are simply counted in a dedicated memory, where all catched elephants are registered. By this way, the algorithm can not only identify and count elephants but also give their size distribution. The relative error on the complementary cumulative distribution function of elephants size is plotted in Figure 3. One can notice that some small elephants are missed because of the refreshment mechanism. Whereas the algorithm gives a very good estimation (with a relative error less than 0.1%)of the number of large elephants (with more than 100 packets).

In Figure 4, the impact of the filling up threshold (r), on the estimation of elephant number is investigated. The best result is obtained for $r = 50\%$. With smaller values of r,($r = 40\%,\ 30\%$), the relative error becomes negative meaning that some elephants are missed. In fact, the memory size being constant, for smaller values of r, the filter is refreshed more frequently. So elephants are loosing more packets. On the contrary, for high values of r, more mice are declared as elephants, and the error generated by the algorithm consists mainly of false positives in this case. So there is clearly a tradeoff for the choice of r.

Figure 5 gives the relative error on the estimated number of elephants for different values of d. The first observation is that, for larger values of d, the algorithm remains very robust even if the following occurs: A flow is declared elephant if its d associated counters reach $C = 20/d$. In a $d = 10$ version, C is only 2. Nevertheless, in stationary regime, the algorithm performs better for $d = 2$. Indeed, it seems on Figure 5 that more elephants are missed for larger d, generating a higher negative error on the estimation of number of elephants. The explanation could be the following: Elephant packets are alternatively incrementing d counters, so when a refreshment occurs, up to d packets of the elephant are lost at the same time, according to the number of packets already

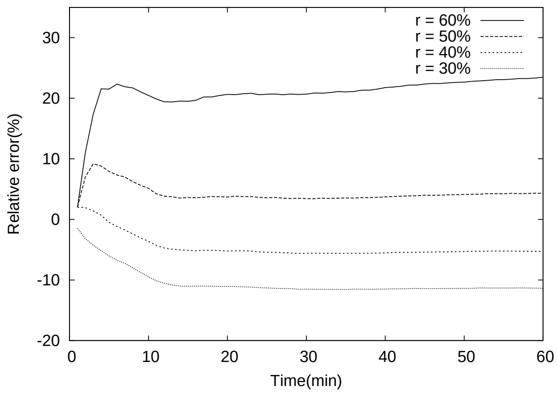

Fig. 4 Impact of r on the estimation of elephants number , $m = 1.05MB$, France Telecom trace

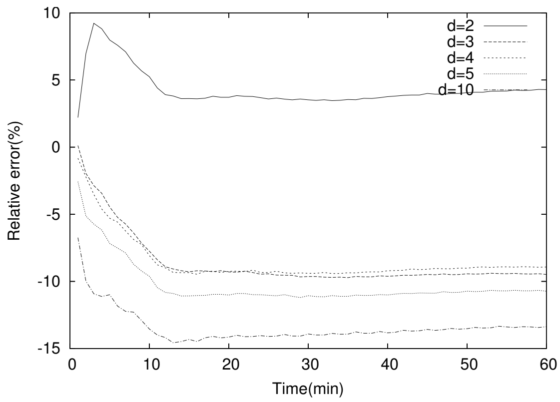

Fig. 5 Impact of d on the estimation of elephants number, $r = 50\%$, France Telecom trace

transmitted. It implies that, if d is large, then elephants with size close to 20 could be more easily lost. In the following, d will remain equal to 2.

Figure 6 presents *the inter-refreshment time* (duration between two successive refreshments in terms of number of arriving packets) for the whole traffic trace. It can be noticed that the stationary phase is reached at the Kth refreshment time. So the transition phase seems to be rather short, according to experiments. The stationary inter-refreshment time using the algorithm based on the supermarket model is higher than the one obtained with the original version of the algorithm. This can be explained by the fact that, with the supermarket scheme, every arriving packet increments exactly one counter, whereas in the original version, if the two selected counters are equal, they are both incremented by one. In particular, when they are both null, they will be both impacted. As a consequence, the proportion of non null counters grows faster and the filling up threshold r is reached more quickly.

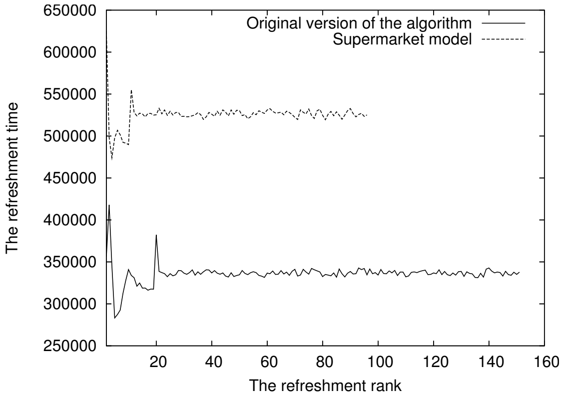

Fig. 6 Duration of the transition and the stationary phase, $r = 50\%$, France Telecom trace

Figure 6 gives an explanation to the behavior of the algorithms plotted in Figure 2. In the original algorithm, the inter-refreshment time is lower thus more elephants are missed. The error is thus negative. In the supermarket version, the error is positive due to false positives.

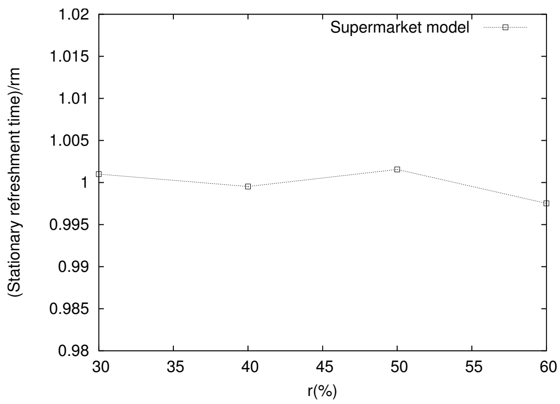

Fig. 7 Comparison between rm and the stationary inter-refreshment time τ_∞^m, France Telecom trace

In Figure 7, the impact of r on the stationary inter-refreshment time τ_∞^m is investigated. More precisely τ_∞^m/rm is plotted for various values of r. According to experiments, τ_∞^m is very close to rm. In fact the refreshment can be seen as removing rm from the sum S of all counters (decreasing by one all non null counters which are exactly rm as the refreshment is performed as soon as the filling up threshold r is reached). In the stationary phase, when m is large, the proportion of counters at i, for $i \in \{0, \dots, C\}$, is w_i at the first order. And just before the next refreshment, rm packets must be inserted into the filter, to let S have its former value. Packets belonging to elephants which have been detected are not taken into account. Those packets are very numerous and they are not inserted into the filter to avoid polluting it. The conclusion is that τ_∞^m/m is close to r even for $C = 10$.

4 Discussion

The performance of the algorithm clearly depends on the filling up r. To have a good estimation of the number of elephants, r must be around 50%. When r has higher values, elephants number will be largely overestimated due to false positives. The key quantity is $\bar{w}(i)$, the stationary proportion of counters at i when m gets large. An explicit expression for $\bar{w}$ is not available even if a numerical value could be computed. Nevertheless, less ambitiously, one can maybe simply find the critical value of r for which $\bar{w}(C)$ gets non negligible. At least, the impact of r on $\bar{w}$ is shown here by simulation.

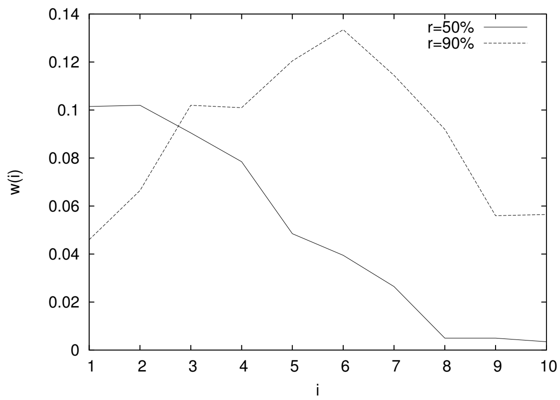

Fig. 8 Impact of r on the limit stationary distribution $\bar{w}$, simulation using only mice with uniform size distribution of mean 4, $m = 2000$

Figure 8 (w is written for $\bar{w}$) is not based on a real traffic trace but on simulation. The objective here is to evaluate the limit stationary distribution $\bar{w}$ if we consider a traffic composed only of mice. The mice mean size is taken equal to four to be close to the real traffic (This value is deduced from the real traffic trace). Under these conditions, we obtain a decreasing limit stationary distribution of $\bar{w}$, when r equals 50%. For a filling up threshold of 90%, counters are very likely to be higher. We can notice that the main part of counters values is around six and there are many counters at C. This explains the fact that, with a filling up threshold around 50%, the algorithm performs better.

The same experiments are now performed on the real traffic trace. Notice that results plotted on figure 9 are very close to those obtained by simulation on figure 8. One can deduce that the elephants present in the real traffic trace do not impact too much the limit stationary distribution $\bar{w}$. As a consequence, to analyze the performance of the algorithm, in particular its generated false positives, flows can be reduced to mice. This is in agreement with the model considered in part II.

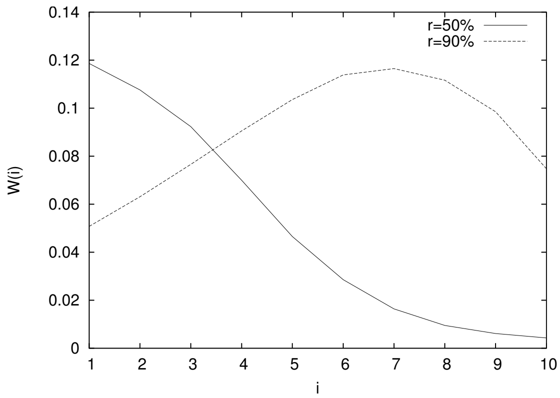

Fig. 9 Impact of r on the limit stationary distribution $\bar{w}$, France Telecom trace

5 Conclusion

In this paper, a new algorithm catching on-line elephants in the Internet is analyzed. This algorithm is based on Bloom filters with a refreshment mechanism that depends on the current traffic intensity. It also uses a conservative way to update counters, called the min-rule. This latter is exactly to increment the lowest counter among a set of d chosen at random, ties being solved at random, as in the supermarket model which provides a much lower tail distribution for the counter values. For a model involving just mice, limit theorems investigate the existence of a deterministic limit for the empirical distribution of counters values, when the filter size gets large. This limit can be exploited to adjust the parameters for the algorithm to perform well. The accuracy of the algorithm and some theoretical results are tested against a traffic trace from France Telecom and by simulations.

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